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Some Considerations on the Application of the Volterra Representation of Nonlinear Networks to Adaptive Echo Cancellers

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In this paper we apply the Volterra representation of nonlinear systems to the echo control problem and propose a generalized adaptive echo canceller which compensates for nonlinear echo paths. We prove that the proposed echo canceller converges and reduces the echo to zero and finally we suggest other applications for the system.

I. INTRODUCTION

The Volterra functional representation of a nonlinear system is a generalization of the well known convolution integral used for linear systems. The validity conditions for the Volterra model are sufficiently weak to be satisfied in many practical applications.

In this paper we apply the Volterra representation to the echo control problem and propose a generalized adaptive echo canceller which compensates for nonlinear echo paths. We prove that the proposed echo canceller converges and reduces the echo to zero in the absence of noise. We also suggest other applications for the adaptive echo canceller presented.

II. THE VOLTERRA FUNCTIONAL REPRESENTATION OF A NONLINEAR 2-PORT

For linear systems it is well known that the impulse response completely determines the input-output relationship. The output signal, $y(t)$, is functionally related to the input, $x(t)$, by the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\, d\tau, \tag{1}$$

where $h(t)$ is the system impulse response. It has been demonstrated by other investigators[1–3] that nonlinear systems, whose outputs do

not depend on the infinite past, obey a more general functional relationship,

$$y(t) = h_0(t) + \int_{-\infty}^{\infty} h_1(\tau_1)x(t - \tau_1)\, d\tau_1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)\, d\tau_1\, d\tau_2$$

$$+ \sum_{n=3}^{\infty} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} x(t - \tau_i)\, d\tau_i . \tag{2}$$

This is an extension of the familiar power series representation of a memoryless nonlinear system and provides for the system to have memory. It is applicable to all nonlinear systems whose outputs depend on the remote past to a vanishingly small extent. For example, it is not applicable to hysterisis and/or switching systems. The terms of (2) were first studied by Volterra[4] and are called Volterra functionals. The kernels, $h_n(\tau_1, \cdots, \tau_n)$, are generally called Volterra kernels.

For the systems that we are concerned with we will assume the following:

(*i*) The zero input response, $h_0(t)$, is identically zero.
(*ii*) The system is causal so that $h_n(\tau_1, \tau_2, \cdots, \tau_n) = 0$ for any $\tau_i < 0$.
(*iii*) The system is stable so that for all n,

$$\underbrace{\int_0^{\infty} \cdots \int_0^{\infty}}_{n} [h_n(\tau_1, \cdots, \tau_n)]^2\, d\tau_1, d\tau_2, \cdots, d\tau_n < \infty .$$

Because of *i* we may rewrite (2) in the form

$$y(t) = \sum_{n=1}^{\infty} y_n(t), \tag{3}$$

where

$$y_n(t) = \underbrace{\int_0^{\infty} \cdots \int_0^{\infty}}_{n} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} [x(t - \tau_i)\, d\tau_i]. \tag{4}$$

III. AN ADAPTIVE ECHO CANCELLER FOR NONLINEAR SYSTEMS

We will now generalize the adaptive echo canceller proposed by Sondhi[5] to cancel the echo of a nonlinear echo path for which the Volterra representation is applicable.

Since the Volterra kernels are square integrable we may represent them by an n dimensional generalized Fourier series,[6–9]

$$h_n(\tau_1, \tau_2, \cdots, \tau_n) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} C_{i_1, i_2, \cdots, i_n} \cdot \Gamma_{i_1}(\tau_1)\Gamma_{i_2}(\tau_2) \cdots \Gamma_{i_n}(\tau_n), \tag{5}$$

where $\{\Gamma_i(t)\}$* is an orthonormal set complete in the $L2$ space. The coefficients of (5) are given by

$$C_{i_1, i_2, \cdots, i_n} = \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} h_n(\tau_1, \tau_2, \cdots, \tau_n) \cdot \Gamma_{i_1}(\tau_1)\Gamma_{i_2}(\tau_2) \cdots \Gamma_{i_n}(\tau_n)\, d\tau_1 \cdots d\tau_n . \tag{6}$$

We will assume that the highest order nonlinearity is of order N. Then substituting (5) into (3) we obtain

$$y(t) = \sum_{n=1}^{N} \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} C_{i_1, i_2, \cdots, i_n} \int_0^{\infty} \Gamma_{i_1}(\tau_1) x(t - \tau_1)\, d\tau_1 \cdot \int_0^{\infty} \Gamma_{i_2}(\tau_2) x(t - \tau_2)\, d\tau_2 \cdots \int_0^{\infty} \Gamma_{i_n}(\tau_n) x(t - \tau_n)\, d\tau_n . \tag{7}$$

Before we proceed it will be convenient to adopt a shorthand notation. We will define

$$\sum_{I_n} (\cdot) \equiv \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} (\cdot), \tag{8}$$

$$C_{I_n} \equiv C_{i_1, i_2, \cdots, i_n} , \tag{9}$$

and

$$w_{i_n} \equiv \int_0^{\infty} \Gamma_{i_n}(\tau_n) x(t - \tau_n)\, d\tau_n .^{\dagger} \tag{10}$$

Thus, (7) becomes

$$y(t) = \sum_{n=1}^{N} \sum_{I_n} C_{I_n} w_{i_1} w_{i_2} \cdots w_{i_n} . \tag{11}$$

Now consider the system shown in Fig. 1. It contains N subsystems designated by the circled numbers. The filters having the set of mutually orthonormal impulse responses $\{\Gamma_{i_1}(t)\}$ and the set of outputs $\{w_{i_1}\}$

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* Some typical sets $\{\Gamma_n(t)\}$ are the impulse responses of tapped delay lines or Laguerre networks.

† Note that w_{i_n} is a function of t although it is not explicitly shown.

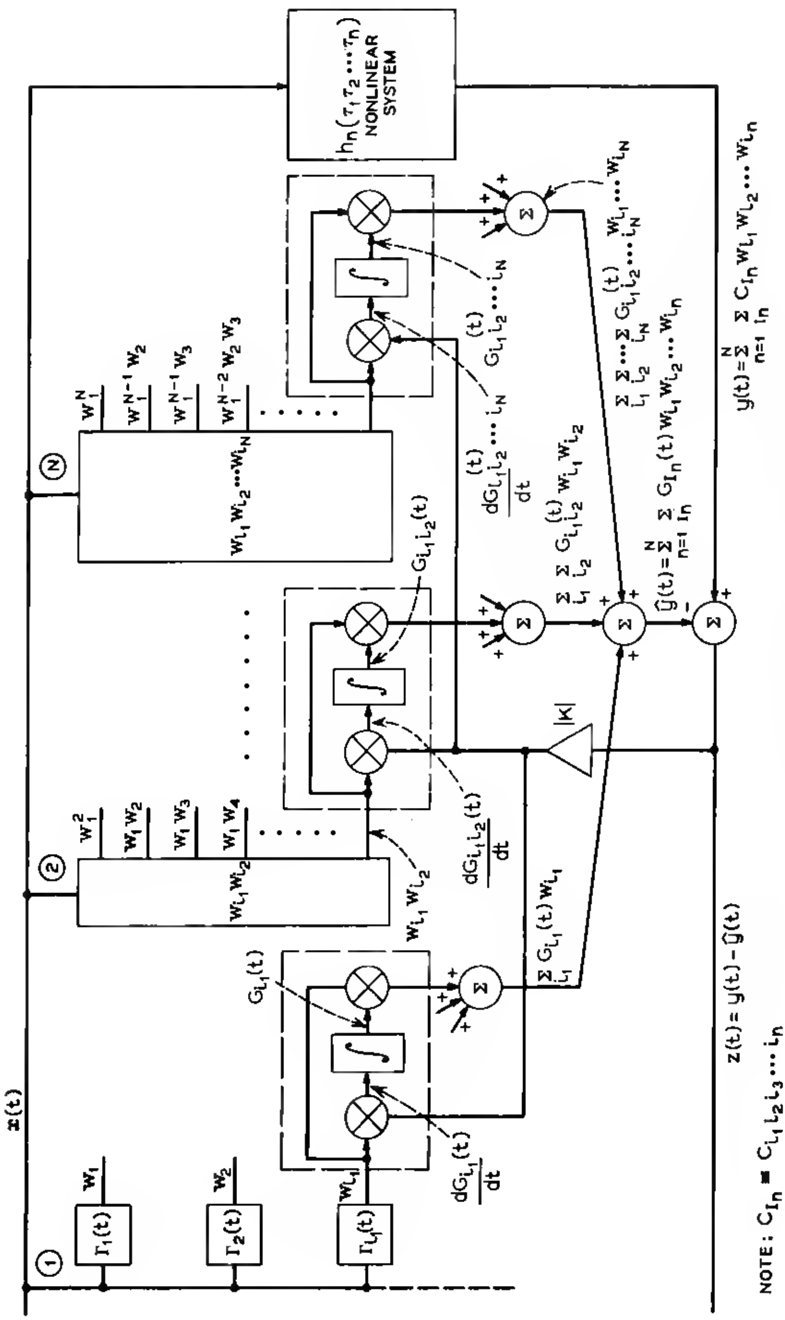

Fig. 1—An adaptive echo canceller for nonlinear systems.

due to the input $x(t)$ are common to all the subsystems. Subsystem 1 produces the output of the filters taken one at a time. Subsystem 2 produces all possible combinations of the product of the filter outputs taken two at a time (i.e., $w_{i_1}w_{i_2}$), and subsystem N produces all possible combinations of the filter outputs taken N at a time (i.e., $w_{i_1}w_{i_2}w_{i_3} \cdots w_{i_n}$). Every output tap of every subsystem has associated with it two multipliers and an integrator connected in the configuration shown in Fig. 1 for only three taps. Other quantities pertinent to the discussion below are also defined in Fig. 1.

We will now show that the tap gains, $G_{I_n}(t)$, of the system converge to the generalized Fourier coefficients C_{I_n} of (6) and, consequently, that $\hat{y}(t)$ converges to $y(t)$ so that the residual echo, $z(t)$, vanishes. We first define

$$R_{I_j}(t) \equiv C_{I_j} - G_{I_j}(t). \tag{12}$$

Then, by inspection, the equation governing the gain of each of the taps of the pth subsystem is

$$\frac{dG_{I_p}(t)}{dt} = |\,K\,|\, w_{i_1}w_{i_2} \cdots w_{i_p} \sum_{n=1}^{N} \sum_{I_n} R_{I_n}(t) w_{i_1}w_{i_2} \cdots w_{i_n}.^{*} \tag{13}$$

From (12) we see that

$$\frac{dR_{I_j}(t)}{dt} = -\frac{dG_{I_j}(t)}{dt}. \tag{14}$$

Also,

$$\frac{dR^2_{I_j}(t)}{dt} = 2R_{I_j}(t)\,\frac{dR_{I_j}(t)}{dt}. \tag{15}$$

Applying (14) and (15) to (13) we obtain

$$\frac{dR^2_{I_p}(t)}{dt} = -2\,|\,K\,|\, R_{I_p}(t) w_{i_1}w_{i_2} \cdots w_{i_p} \sum_{n=1}^{N} \sum_{I_n} R_{I_n}(t) w_{i_1}w_{i_2} \cdots w_{i_n}. \tag{16}$$

Summing over all the taps within the pth subsystem, and over the N subsystems, yields

$$\frac{d}{dt} \sum_{p=1}^{N} \sum_{I_p} R^2_{I_p}(t) = -2\,|\,K\,|\left(\sum_{n=1}^{N} \sum_{I_n} R_{I_p}(t) w_{i_1}w_{i_2} \cdots w_{i_n} \right)^2. \tag{17}$$

Note that the right-hand side of (17) is always negative or zero.

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* Where the set $(i_1 i_2 \cdots i_p)$ are the components of I_p.

Also note that

$$\psi(t) \equiv \sum_{p=1}^{N} \sum_{I_p} R_{I_p}^2(t) \geqq 0. \tag{18}$$

Thus, we have a nonnegative function, whose derivative is always nonpositive, and conclude that $\psi(t)$ must be nonincreasing. It is strictly decreasing whenever

$$\sum_{n=1}^{N} \sum_{I_n} R_{I_n}(t) w_{i_1} w_{i_2} \cdots w_{i_n} \not\equiv 0.$$

Since, in general

$$w_{i_1} w_{i_2} \cdots w_{i_n} \not\equiv 0,$$

it is clear that

$$\lim_{t \to \infty} \psi(t) = 0. \tag{19}$$

Since every term in $\psi(t)$ is positive, we further conclude that

$$\lim_{t \to \infty} R_{I_p}(t) = 0 \quad \text{for} \quad p = 1, 2, \cdots, N. \tag{20}$$

Applying (20) to (12) we see that

$$\operatorname*{limit}_{t \to \infty} G_{I_p}(t) = C_{I_p}; \qquad p = 1, 2, \cdots, N. \tag{21}$$

Thus we have shown that the tap gains converge to the generalized Fourier coefficients of $h_n(\tau_1, \cdots, \tau_n)$, which is what we set out to prove.

The reader can no doubt appreciate that solving (17) for anything but the final value would be difficult and would only be valid for a specific $x(t)$. The solution for even a very simple input signal may not be tractable. We will not attempt such an analysis here. However, from (17) we see that the larger we make $|K|$ the quicker the echo vanishes. However, $|K|$ cannot be made arbitrarily large because doubling talking periods* will cause divergence problems.

IV. EFFECTS OF CIRCUIT NOISE AND OTHER CONSIDERATIONS

The previous analysis assumes a noiseless output signal $y(t)$. When circuit noise, $n(t)$, generated in the echo path is present equation (17) would be written as

* Doubling talking is said to occur when both speakers in a telephone conversation are speaking simultaneously.

$$\frac{d}{dt}\sum_{j=1}^{N}\sum_{I_p} R_{I_p}^2(t) = -2\,|\,K\,|\left(\sum_{n=1}^{N}\sum_{I_n} R_{I_n}(t) w_{i_1} w_{i_2}\cdots w_{i_n}\right)^2$$
$$- 2\,|\,K\,|\,n(t)\left(\sum_{n=1}^{N}\sum_{I_n} R_{I_n}(t) w_{i_1} w_{i_2}\cdots w_{i_n}\right).$$

By choosing $|K|$ small so that the rate of change of the G_{I_n} is small, it is reasonable to assume that $n(t)$ is statistically independent of all other variables in the equation. Then, assuming $n(t)$ has zero mean, upon taking the ensemble average of both sides of the above equation we obtain

$$\frac{d}{dt}\left(\left\langle\sum_{j=1}^{N}\sum_{I_p} R_{I_p}^2(t)\right\rangle\right)$$
$$= -2\,|\,K\,|\left\langle\left(\sum_{n=1}^{N}\sum_{I_n} R_{I_n}(t) w_{i_1} w_{i_2}\cdots w_{i_n}\right)^2\right\rangle. \tag{22}$$

The same argument used before then yields

$$\lim_{t\to\infty}\left\langle\sum_{j=1}^{N}\sum_{I_p} R_{I_p}^2(t)\right\rangle = 0. \tag{23}$$

Of course, before this limit is reached our statistically independent assumption would no longer be valid, but at least we may conclude that convergence will take place until the noise power becomes comparable to the echo power.

There are several other difficulties associated with the echo canceller. Recall that we represented the nth-order Volterra kernel by an "n" fold infinite summation [equation (5)]. Obviously, if such a system is to be implemented it cannot include an infinite number of elements and (5) can only be approximated by

$$h_n(\tau_1\,,\,\cdots\,,\,\tau_n) \approx \sum_{i_1}^{J_1}\sum_{i_2}^{J_2}\cdots\sum_{i_n}^{J_2} G_{i_1,i_2,\ldots,i_n}\Gamma_{i_1}(\tau_1)\Gamma_{i_2}(\tau_2)\cdots\Gamma_{i_n}(\tau_n), \tag{24}$$

where the J's are finite and as small as possible for an adequate description.

Furthermore, even if the J's are small the system shown in Fig. 1 requires a large number of components. However, one point may not be obvious. It can be shown that the Volterra kernels are symmetrical.[2] As a result it may be verified that all the $C_{i_1 i_2\cdots i_n}$ are equal for any permutation of a specific set of numbers i_1, i_2, $\cdots$, i_n. For example,

$$C_{123} = C_{231} = C_{312} = C_{213}, \text{ etc..}$$

Since the set of gains, $G_{i_1 \cdots i_n}$, for any combination of a specific set of integers $i_1 \cdots i_n$ are the result of the same physical operations, only one tap is required to account for all such terms. For example, consider the w_1w_2 tap and the w_2w_1 tap. From Fig. 1,

$$G_{12} = \int w_1w_2[y(t) - \hat{y}(t)]\, dt,$$

$$G_{21} = \int w_2w_1[y(t) - \hat{y}(t)]\, dt.$$

It is obvious that

$$G_{21} = G_{12}.$$

The contribution of G_{12} and G_{12} taps toward $\hat{y}(t)$ is given by

$$\hat{y}(t)\,|_{12} = G_{12}w_1w_2 + G_{21}w_2w_1 = (G_{12} + G_{21})w_{12} = 2G_{12}w_{12}.$$

Thus, the G_{12} and G_{21} taps may be replaced by a single tap. Any set of taps, $G_{i_1 \cdots i_n}$, for any permutation of a specific set of numbers, $i_1 \cdots i_n$, can be replaced by a single tap. This considerably reduces the total number of taps required.

V. OTHER APPLICATIONS

Although we have stressed the echo cancellation application, these ideas may also be useful in other areas. Two possible uses are described below.

5.1 *Compensator for a Nonlinear System*

Assume that we wish to linearize the nonlinear system shown in Fig. 1, such that the resulting output, $z(t)$, can be expressed by the linear convolution integral,

$$z(t) = \int_0^\infty h_1(\tau_1)x(t - \tau_1)\, d\tau_1 \,. \tag{25}$$

This can be done by first allowing the adaptive system of Fig. 1 to converge long enough so that the members of the set $\{G_{I_j}(t)\}$ can be considered to equal the corresponding members of the set $\{C_{I_j}\}$ [equation (21)]. After convergence, the members of the set $\{G_{I_1}\}$ are forced to zero while the members of the sets $\{G_{I_j}\}$, $j \neq 1$ are fixed at the values determined previously. As a result the compensated output will satisfy equation (25).

5.2 *Nonlinear System Synthesizer*

Suppose one wishes to study the electrical characteristics of a nonlinear system which cannot be brought into the laboratory. He could do this by making input/output tape recordings of the system, and use these as an input to a computer simulation of the adaptive system of Fig. 1. A good choice of an input signal would be white noise or any other easily generated broadband signal. By allowing the simulation to converge and then fixing the tap gains G_{l_n} at their final value, the nonlinear characteristic can be identified. Then it can be determined how the field system will behave for any arbitrary input by applying this input to the computer simulation with the tap gains fixed at the values determined previously.

VI. SUMMARY AND CONCLUSIONS

The Volterra representation is a concise method of characterizing nonlinear systems with memory when the output does not depend on the infinite past. It is a generalization of the convolution integral used in linear analysis, and many familiar concepts may be extended. Using the Volterra representation, we have proposed a generalized adaptive echo canceller capable of nonlinear compensation. A disadvantage of the proposed echo canceller is its complexity. The pursuit of these ideas would be greatly enhanced if an efficient means of measuring the Volterra kernels of an unknown network could be found. The author and other researchers are presently working toward this end and several ideas have been proposed. To date, unfortunately, they all suffer from the complexity problem.

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